NOTES FINANCIAL MATHEMATICS

APPLICABLE FORMULAE FROM FORMULA SHEET

$A = P(1+ni)$ $A = P(1-ni)$ $A = P(1-i)^n$ $A = P(1+i)^n$

$F = \frac{x[(1+i)^n - 1]}{i}$ $P = \frac{x[1-(1+i)^{-n}]}{i}$

	Formulae	Type	Common terminology
	INCREASING AMOUNTS		
5.1Investments / Growth	5.1.1 $A = P(1+i.n)$	**Simple interest**	- rate per annum at simple interest
	5.1.2 $A = P(1+i)^n$	**Compound interest,** lump sum over a period of time	-Interest **compounded** yearly/annually, monthly, quarterly or half yearly -invest money, deposit money, grow, appreciate, - compound increase/ compound growth/ inflation
	5.1.3 $Fv = \frac{x[(1+i)^n - 1]}{i}$	**Annuities**, same deposits / instalments at equal intervals e.g monthly, yearly, quarterly or half yearly etc.	-monthly, yearly, quarterly or half-yearly instalment
	DECREASING AMOUNTS		
5.2 Depreciation / Decay / Loss of value	5.2.1 $A = P(1-i.n)$	**Straight line** depreciation	-straight-line depreciation/ linear reduction/ straight line decay;
	5.2.2 $A = P(1-i)^n$	Depreciation on a **reducing**/diminishing **balance**	- value depreciates at x % p.a - reducing balance/ diminishing balance/ compound decay/ compound decrease;
5.3.Loan Repayments	5.3.1 $Pv = \frac{x[1-(1+i)^{-n}]}{i}$	Loan repayment Or Getting monthly payments from an investment amount that will decrease	-loan repayment - making × monthly, quarterly, or yearly instalment, etc -deposits made at the end of every month, year, quarter or half-year -someone borrows the money
	INTEGRATION OF FORMULAE		
	5.3.2 $P(1+i)^n = \frac{x[(1+i)^n - 1]}{i}$	Balance on outstanding loan based on monthly repayments. This can also be done in two separate steps.	

5.4 Effective	5.4.1 $1 + i_{eff} = \left(1 + \frac{i}{n}\right)^n$	nominal / effective rate	nominal to effective rate

EXAMPLE 1

David invests R50 000 at 8% p.a. compound interest. What will his money grow to in 5 years if interest is compounded

(a) annually? (b) quarterly? (c) monthly?

Solution

(a) $A = P(1+i)^n$

$= 50\,000(1+0{,}08)^5$

$= R73\,466{,}40$

(b) Since interest is calculated quarterly, we need to calculate how many quarters there are in a 5 years: $n = 5 \times 4 = 20$. (There are 4 quarters in a year.)

We must also divide the interest rate by 4 to get a quarterly rate: $i = \frac{0{,}08}{4} = 0{,}02$.

$A = P(1+i)^n$

$= 50\,000(1+0{,}02)^{20}$

$= R74\,297{,}37$

(c) $n = 5 \times 12 = 60 \quad i = \frac{0{,}08}{12}$

$A = P(1+i)^n$

$= 50\,000\left(1+\frac{0{,}08}{12}\right)^{60}$

$= R74\,492{,}29$

EXAMPLE 6

The value of a vehicle worth R200 000 depreciates at 12% p.a. Calculate the value of the vehicle in 5 years' time if depreciation is calculated
(a) using the straight line method. (b) on the reducing balance.

Solution

(a)
$$\begin{aligned} A &= P(1-in) \\ &= 200\,000(1-0{,}12\times 5) \\ &= R80\,000 \end{aligned}$$

(b)
$$\begin{aligned} A &= P(1-i)^n \\ &= 200\,000(1-0{,}12)^5 \\ &= R105\,546{,}38 \end{aligned}$$

EXAMPLE 7

Sandy buys a printer for R4 000 and sells it 3 years later for R2 200. Calculate the annual rate of depreciation if the straight line method is used.

Solution

$$A = P(1-in)$$
$$\therefore 2\,200 = 4\,000(1-i\times 3)$$
$$\therefore \frac{2\,200}{4\,000} = 1-3i$$
$$\therefore 3i = 1-\frac{2\,200}{4\,000}$$
$$\therefore i = \left(1-\frac{2\,200}{4\,000}\right)\div 3 = 0{,}15$$

The annual rate of depreciation is 15%

INFLATION

We all know that prices increase over time. This is called *inflation*. We calculate inflation using the same formula as for compound interest. Inflation is usually calculated annually.

EXAMPLE 8

A loaf of white bread currently costs R14. How much will a similar loaf of bread cost 5 years from now if an annual inflation rate of 7,5% is maintained?

Solution

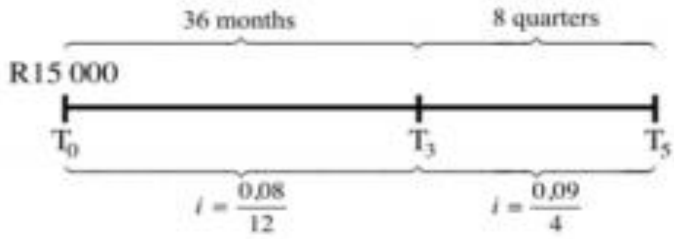
$$\begin{aligned} A &= P(1+i)^n \\ &= 14(1+0{,}075)^5 \\ &= R20{,}10 \end{aligned}$$

CHANGE IN INTEREST RATE

EXAMPLE 9

Sibusiso deposits R15 000 into his bank account and leaves it in the account for 5 years. For the first 3 years interest is calculated at 8% p.a. compounded monthly. Thereafter the interest rate changes to 9% p.a. compounded quarterly. How much money will he have in his account at the end of the 5 years?

Solution



Move the R15 000 forward, from T_0 to T_5 :

$$15\,000\left(1+\frac{0,08}{12}\right)^{36}\cdot\left(1+\frac{0,09}{4}\right)^{8}$$

$= R22\ 765,78$

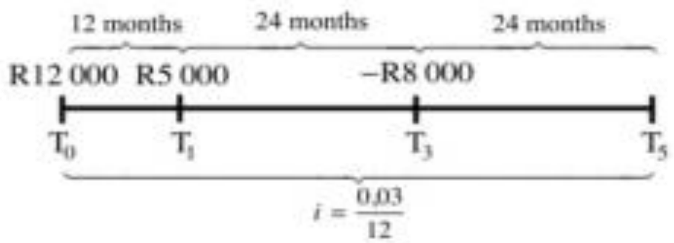
MULTIPLE PAYMENTS/WITHDRAWALS

When multiple amounts of money are involved, each amount can be moved around on the timeline separately. This way complicated calculations are simplified greatly.

EXAMPLE 10

Cheryl's bank offers 3% p.a. compounded monthly on her savings account. She opens the account with a deposit of R12 000. A year later, she deposits another R5 000. Another 2 years later she withdraws R8 000. How much money will be in her bank account 5 years after the account was opened?

Solution



Move all the money forward to T_5 :

(R12 000 from T_0 to T_5, R5 000 from T_1 to T_5, and – R8 000 from T_3 to T_5)

$$12\,000\left(1+\frac{0,03}{12}\right)^{60}+5\,000\left(1+\frac{0,03}{12}\right)^{48}-8\,000\left(1+\frac{0,03}{12}\right)^{24}$$

$= R11\,081,99$

EXAMPLE 3

Mokgeseng wants to save up money for an overseas vacation after he graduates from university. He opens a savings account and immediately deposits R500. After this he continues to deposit R500 at the end of each month for 4 years. The interest rate remains fixed at 8% p.a. compounded monthly.

(a) How much will he have in the savings account at the end of the 4 years?

(b) At the end of the 4 years, Mokgeseng stops making payments. He leaves the money in the account for another 2 years, while he completes a postgraduate degree. How much money will he have in the account at the end of these two years?

Solution

(a)

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$=\frac{500\left[\left(1+\frac{0{,}08}{12}\right)^{49}-1\right]}{\frac{0{,}08}{12}}$$

$$=\text{R}28\ 862{,}79$$

(b)

$$A=P(1+i)^n$$

$$=28\ 862{,}79\left(1+\frac{0{,}08}{12}\right)^{24}$$

$$=\text{R}33\ 852{,}82$$

EXAMPLE 5

Linda wants to save up R30 000 to buy a new gaming laptop. She can afford to save R3 000 every six months. Interest is calculated at 10,2% p.a. compounded semi-annually. How many payments will she have to make to have at least R30 000 in savings?

Solution

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore 30\ 000 = \frac{3\ 000\left[\left(1+\frac{0,102}{2}\right)^n - 1\right]}{\frac{0,102}{2}}$$

$$\therefore \frac{30\ 000 \times \frac{0,102}{2}}{3\ 000} = \left(1+\frac{0,102}{2}\right)^n - 1$$

$$\therefore \frac{51}{100} = \left(1+\frac{0,102}{2}\right)^n - 1$$

$$\therefore \frac{151}{100} = \left(\frac{1\ 051}{1\ 000}\right)^n$$

$$\therefore n = \log_{\frac{1\ 051}{1\ 000}}\left(\frac{151}{100}\right)$$

$$\therefore n = 8,28$$

She will have to make 9 payments to have at least R30 000 in savings.

EXAMPLE 6

Mpho can afford to pay back at most R8 000 per month on a car loan. The interest on a car loan is 15% p.a. compounded monthly. He will make his first payment one month from now and will repay the loan over a period of 5 years. What is the price of the most expensive car Mpho can afford?

Solution

$$P = \frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$= \frac{8\ 000\left[1-\left(1+\frac{0,15}{12}\right)^{-60}\right]}{\frac{0,15}{12}}$$

$$= R336\ 276,73$$

EXAMPLE 7

Jenny wishes to repay a loan of R150 000, by means of 16 equal quarterly payments, starting three months from now. The interest rate on the loan is 21,5% p.a. compounded quarterly.

(a) Calculate what Jenny's quarterly payment will be.

(b) Calculate the total interest that Jenny will pay on the loan.

Solution

(a) Notice that 3 months = 1 quarter, which means payments start one period after the loan was made as is normally the case.

$$P = \frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$\therefore 150\ 000 = \frac{x\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]}{\frac{0,215}{4}}$$

$$\therefore \frac{150\ 000 \times \frac{0,215}{4}}{\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]} = x$$

$$\therefore x = \text{R}14\ 212,35$$

(b) Total Repayments = 14 212,35 × 16 = R227 397,60

Interest = Total Repayments – Loan Value

= R227 397,60 – R150 000

= R77 397,60

$$P=\frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$\therefore 150\ 000=\frac{x\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]}{\frac{0,215}{4}}$$

$$\therefore \frac{150\ 000\times\frac{0,215}{4}}{\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]}=x$$

$$\therefore x=\text{R}14\ 212,35$$

(b) Total Repayments $=14\ 212,35\times 16=\text{R}227\ 397,60$

Interest = Total Repayments – Loan Value

$=\text{R}227\ 397,60-\text{R}150\ 000$

$=\text{R}77\ 397,60$

EXAMPLE 8

Karel has to pay off a loan of R75 000. He can afford to pay R1 500 per month. The interest rate is 16,2% p.a. compounded monthly. How many payments will he have to make?

Solution

$$P=\frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$\therefore 75\ 000=\frac{1\ 500\left[1-\left(1+\frac{0,162}{12}\right)^{-n}\right]}{\frac{0,162}{12}}$$

$$\therefore \frac{75\ 000\times\frac{0,162}{12}}{1\ 500}=1-\left(1+\frac{0,162}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,162}{12}\right)^{-n}=1-\frac{75\ 000\times\frac{0,162}{12}}{1\ 500}$$

$$\therefore \left(\frac{2\ 027}{2\ 000}\right)^{-n}=\frac{13}{40}$$

$$\therefore -n=\log_{\frac{2\ 027}{2\ 000}}\frac{13}{40}$$

$$\therefore n=-\log_{\frac{2\ 027}{2\ 000}}\frac{13}{40}=83,81$$

He will make 84 payments, consisting of 83 payments of R1500 and one lesser payment*.

*In the next section you will learn how to calculate the value of this lesser payment.

EXAMPLE 9

Sibusiso took out a loan of R250 000. He makes monthly payments of R5 000 on the loan and interest is calculated at 15% p.a. compounded monthly. What is the outstanding balance on the loan after 3 years (directly after the payment made at the end of the 3rd year)?

Solution

$$\text{OB} = \text{P}(1+i)^{m} - \frac{x\left[(1+i)^{m}-1\right]}{i}$$

$$\therefore \text{OB} = 250\,000\left(1+\frac{0,15}{12}\right)^{36} - \frac{5\,000\left[\left(1+\frac{0,15}{12}\right)^{36}-1\right]}{\frac{0,15}{12}}$$

$$\therefore \text{OB} = \text{R}165\,408,43$$

THE LAST PAYMENT

When the payment amount is fixed beforehand, it is usually not possible to settle the loan by an integer number of equal payments. In such cases, there will be a last payment, smaller than the others, to settle the loan. This payment is made one period after the last full payment:

$$\text{Last payment} = \text{Outstanding balance after the last full payment} \times (1+i)$$

EXAMPLE 10

George has to repay a loan of R375 000. The interest rate is 14% p.a. compounded monthly. He pays back R7 500 per month.

(a) How many payments will George have to make? (b) What will his last payment be?

Solution

(a) $$375\,000 = \frac{7\,500\left[1-\left(1+\frac{0,14}{12}\right)^{-n}\right]}{\frac{0,14}{12}}$$

$$\therefore \frac{375\,000\times\frac{0,14}{12}}{7\,500} = 1-\left(1+\frac{0,14}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,14}{12}\right)^{-n} = 1-\frac{375\,000\times\frac{0,14}{12}}{7\,500}$$

$$\therefore \left(\frac{607}{600}\right)^{-n} = \frac{5}{12}$$

$$\therefore -n = \log_{\frac{607}{600}}\left(\frac{5}{12}\right)$$

$$\therefore n = -\log_{\frac{607}{600}}\left(\frac{5}{12}\right) = 75,47706563$$

He will make 76 payments (75 payments of R7 500 and 1 lesser payment).

(b) Calculate the outstanding balance after the last full payment (the 75th payment):

$$OB = P(1+i)^m - \frac{x\left[(1+i)^m - 1\right]}{i}$$

$$\therefore OB = 375\,000\left(1+\frac{0,14}{12}\right)^{75} - \frac{7\,500\left[\left(1+\frac{0,14}{12}\right)^{75} - 1\right]}{\frac{0,14}{12}}$$

$$\therefore OB = R3\,547,457499$$

$$\text{Last Payment} = 3547,457499 \times \left(1+\frac{0,14}{12}\right)$$

$$= R3\,588,84$$